Learning Objective: Understand what the following means, and why it is true.

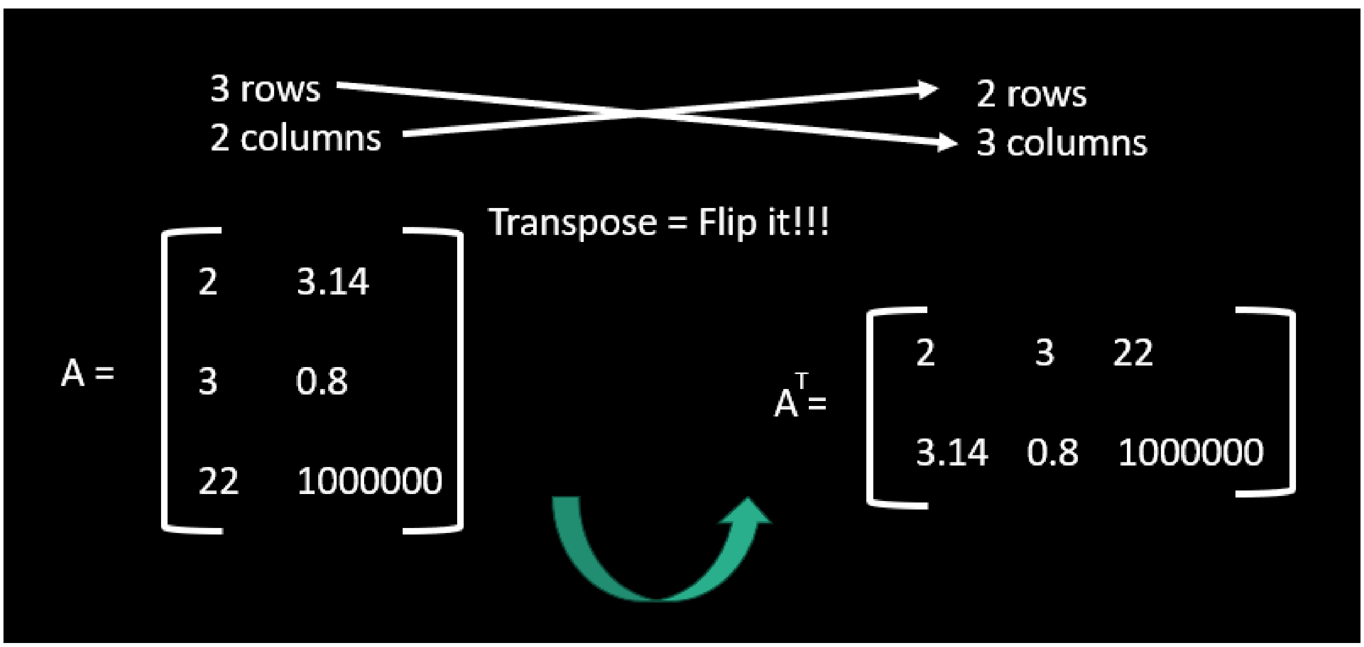
- $\mathcal{R}(A)$ is orthogonal to $\mathcal{N}(A^\top)$

What does this even mean???

- What is A?
- What is $A^\top$ (A-transpose)?
- What is $\mathcal{R}$?
- What is $\mathcal{N}$?
- What is orthogonal?

What are A and $A^\top$?

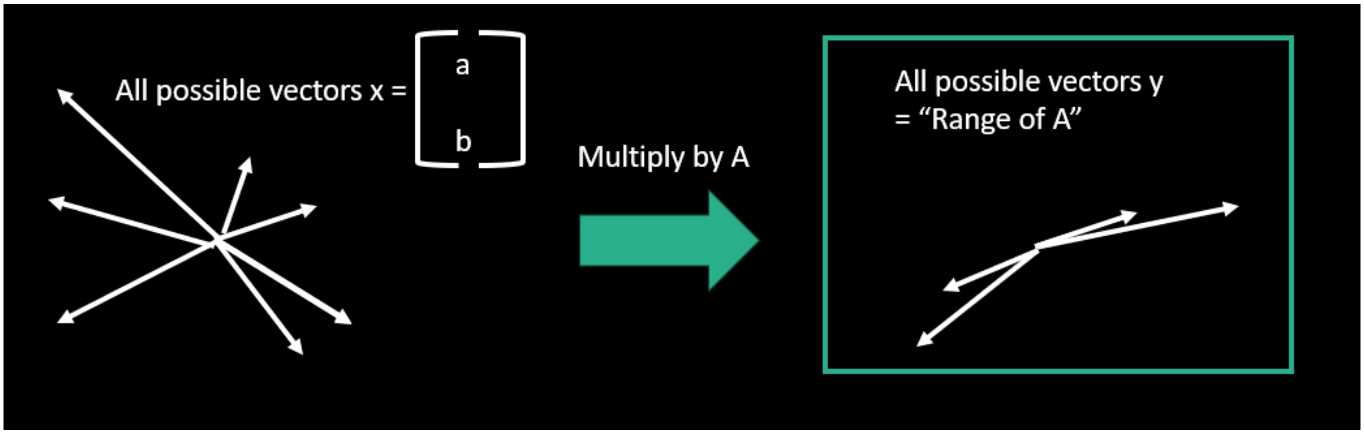
A is a matrix, rows and columns of numbers



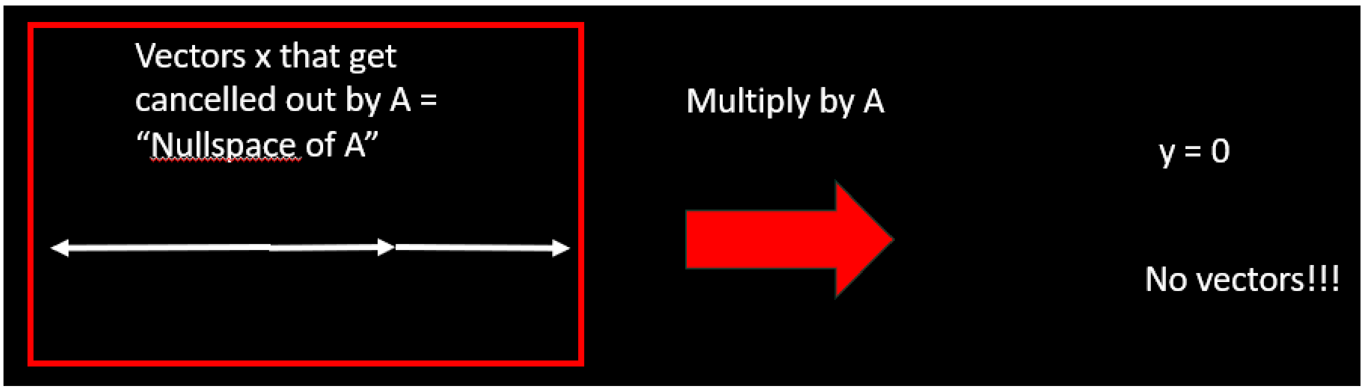
What are $\mathcal{R}(A)$ and $\mathcal{N}(A)$?

$\mathcal{R}(A)$ = "The Range of A"
- Consider the matrix multiplication $y = Ax$.

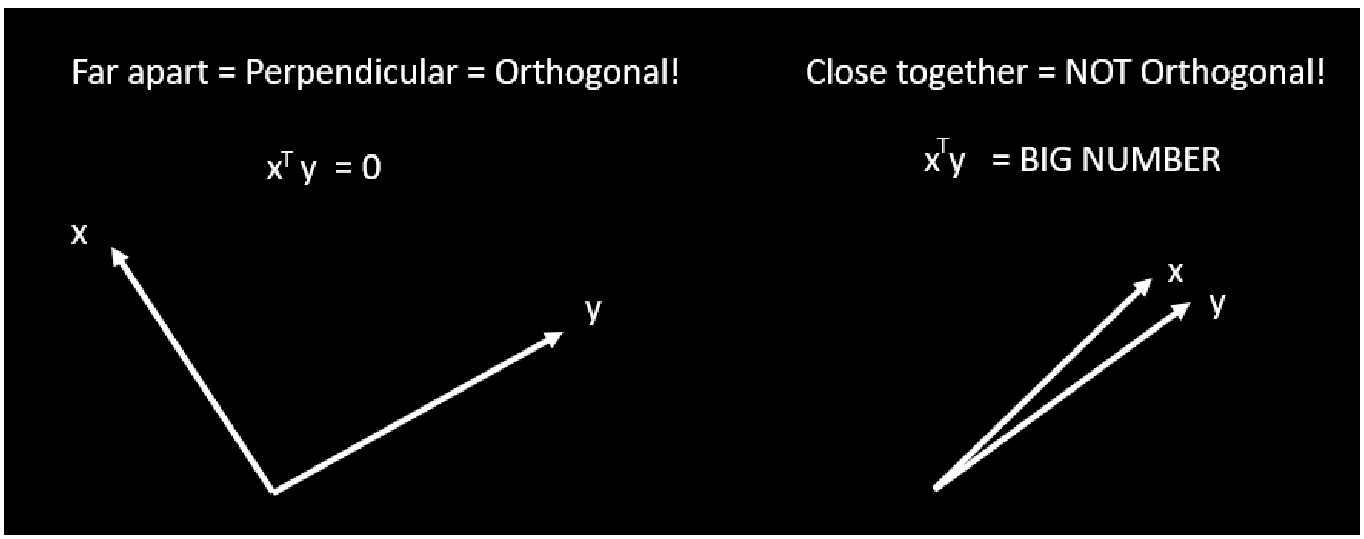
- For ALL POSSIBLE x, (yes, ALL POSSIBLE), the "range of A" is all possible y



$\mathcal{N}(A)$ = "The Nullspace of A"



What is orthogonal?



Two vectors x and y are orthogonal if their DOT PRODUCT is 0!

- $x^\top y = y^\top x = 0$: ORTHOGONAL
- $x^\top y = y^\top x = 1$: NOT ORTHOGONAL

$\mathcal{R}(A)$ is orthogonal to $\mathcal{N}(A^\top)$

- Let's say x is in the nullspace of $A^\top$

- $y = A^\top x = 0$
- $y = 0$
- The dot product between y and any vector is 0!
- $z^\top y$ = $z^\top (A^\top x)$ = 0

The big trick

- Flip it around, it will be equal!
- $z^\top y$ = $y^\top z$
- $(x^\top A)z = x^\top (Az)$
- Set $w = Az$
- $x^\top w$ = 0
- x and w are always orthogonal
- x is in $\mathcal{N}(A^\top)$
- w is in $\mathcal{R}(A)$
- QED

$\mathcal{R}(A)$ is orthogonal to $\mathcal{N}(A^\top)$

You can show the opposite: $\mathcal{R}(A^\top)$ is orthogonal to $\mathcal{N}(A)$