$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}_{2\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \begin{matrix} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 \\ b_{21}x_1 + b_{22}x_2 + b_{23}x_3 \end{matrix}$$

$b_1 x$ $-B$

$b_2 b$ $\begin{bmatrix} -x(T)^T \\ -x(T)^T \end{bmatrix}$

First fit B, then A

$$e = y^* - Bx(T) = y^* - \left(x(T)^T B^T\right)^T \quad {}_{1\times 3} \; {}_{3\times 2}$$

$$\frac{\partial e}{\partial B} = -x(T)^T$$

$$\sqrt{\left(y_1^* - b_{11}x_1 - b_{12}x_2 - b_{13}x_3\right)^2 + \left(y_2^* - b_{21}x_1 - b_{22}x_2 - b_{23}x_3\right)^2}$$

$$\boxed{1.\ B' = B + h\left(\begin{bmatrix} x(T)^T \end{bmatrix}\right)} \text{ if } \frac{1}{2\sqrt{\quad}}$$

repeat many times until the error is lowered

$\frac{\partial e}{\partial b_{11}}, \frac{\partial e}{\partial b_{12}}, \dots$ find all of these

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next $\quad y_1^* + \left(-b_{11}k_{11} - b_{12}k_{21} - \dots\right)e^{\lambda_1 t} + \left(-b_{11}k_{12} - b_{12}k_{22} - \dots\right)e^{\lambda_2 t} + \dots$

SVD

Linear functions can be decomposed into a scaling part and a rotating part, and it is often the case that the scaling part is more useful and interesting because the scaling part exposes the skeleton of the transform. The eigendecomposition expresses this idea effectively, where, given a square matrix $A^{N\times N}$, where $Av_i = \lambda_i v_i$, $i \in [1, N]$,

$$A = V \Lambda V^{-1}, \quad V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_N \\ | & | & & | \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_N \end{bmatrix}$$

☆ (If you're not familiar w/ change of basis, don't continue) (Ignore Jordan edge case)

However, you might observe that V is not always a pure rotation. There is no guarantee that eigenvectors are all orthogonal to one another. This means that we don't have a clean split between the scaling and rotation parts of the transform A. Is there a way to express A as change of basis where the transform to the scaling domain is a pure rotation?

In order to explore this question more, let's entertain the scenario where V is a rotation. This implies that the vectors in V are all orthogonal, which means $V^TV = I$ and $V^T = V^{-1}$, so instead of $A = V\Lambda V^{-1}$, we get $A = V\Lambda V^T$. If we assume V is a rotation, or a unitary matrix, what must A look like? It turns out that A is a symmetric matrix, where $A = A^T$.
(To convince yourself this is the case, first prove why a matrix VV^T (v doesn't have to be a rotation) is symmetric. Then it becomes easier to see why $V\Lambda V^T$ is symmetric).

Now we can go the opposite direction and see that if A happens to be symmetric, we can get an eigendecomposition that looks like $A = V\Lambda V^T$, hence getting our scale+rotate form.

However, does this mean that if A is not symmetric, all hope is lost? To explore this question, we can start by realizing that we can use A to generate symmetric matrices. If we can conjure out symmetric matrices, that could make it easier to see rotation components.

Set $M = A^TA$ and $\Sigma = AA^T$.

Now we have two symmetric matrices of interest, and they can take the form $M = V_M \Lambda_M V_M^T$ and $\Sigma = V_\Sigma \Lambda_\Sigma V_\Sigma^T$.

Since eigenvalues stay the same after a transpose, $\Lambda_M = \Lambda_\Sigma$

$$\left(\begin{aligned} A^TA = \lambda v \rightarrow (AA^T)A &= \lambda Av \\ A(A^TA) &= \lambda Av \end{aligned} \rightarrow \text{shared eigenvalues} \right)$$

$M = A^T A = V_M \Lambda_M V_M^T$

$\Sigma = AA^T = V_\Sigma \Lambda_M V_\Sigma^T$

(✱ the relationship between A^TA and AA^T is not trivial!)

If we stare at this long enough and try to find a relationship between the V matrices and A, you might find it interesting that if you split Λ_M so that

$$M = \left(V_M \Lambda_M^{1/2}\right)\left(\Lambda_M^{1/2} V_M^T\right) \quad \Sigma = \left(V_\Sigma \Lambda_M^{1/2}\right)\left(\Lambda_M^{1/2} V_\Sigma^T\right)$$

Now M and Σ have two halves that are symmetric to one another! $\left(V_M \Lambda_M^{1/2}\right)^T = \Lambda_M^{1/2} V_M^T$

Since M and Σ are constructed by multiplying A with its transpose, it is super tempting at this point to declare

$$\cancel{A = \Lambda_M^{1/2} V_M^T = V_\Sigma \Lambda_M^{1/2}}$$

Alas, this is not the case. This is because given a symmetric matrix $M = B^T B$, if we try and find B you'll see that B is not unique! So maybe B=A, or it doesn't. We can find a lot of matrices that can construct M, such as $\Lambda_M^{1/2} V_M^T$, and it doesn't need to equal A.

The non-uniqueness of B can be expressed as

$$M = \left(V_M \Lambda_M^{1/2} U_1^T\right)\left(U_1 \Lambda_M^{1/2} V_M^T\right) \quad \Sigma = \left(V_\Sigma \Lambda_M^{1/2} U_2^T\right)\left(U_2 \Lambda_M^{1/2} V_\Sigma^T\right)$$

$$U_1^T U_1 = U_2^T U_2 = I$$

$$B_1 = V_M \Lambda_M^{1/2} U_1^T \qquad B_2 = V_\Sigma \Lambda_M^{1/2} U_2^T$$

The U matrices cancel out in the middle, and $B_{1,2}$ still satisfy the conditions $M = B_1 B_1^T$ $\Sigma = B_2 B_2^T$ to create the same symmetric matrices. U can be any rotation!

So how are we supposed to find $U_{1,2}$ such that
$$A = U_1 \Lambda_M^{1/2} V_M^T = V_\Sigma \Lambda_M^{1/2} U_2^T \quad ?$$

Since we have two unknowns U_1 and U_2 and 2 constraint equations $M = A^T A = V_M \Lambda_M V_M^T$ $\Sigma = AA^T = V_\Sigma \Lambda_M V_\Sigma^T$, we have enough information to solve for U_1 and U_2.

$$M = A^T A = \left(V_M \Lambda_M^{1/2} U_1^T\right)\left(U_1 \Lambda_M^{1/2} V_M^T\right) \qquad \Sigma = AA^T = \left(V_\Sigma \Lambda_M^{1/2} U_2^T\right)\left(U_2 \Lambda_M^{1/2} V_\Sigma^T\right)$$

Switch

$$\Sigma = AA^T = \left(U_1 \Lambda_M^{1/2} \underbrace{V_M^T\right)\left(V_M}_{\text{unitary, cancel}} \Lambda_M^{1/2} U_1^T\right)$$ invoke eigendecomp of Σ

$$= U_1 \Lambda_M U_1^T = V_\Sigma \Lambda_M V_\Sigma^T$$

$$\boxed{U_1 = V_\Sigma}$$

Use the same logic to find $\boxed{U_2 = V_M}$

Therefore, $\boxed{A = V_\Sigma \Lambda_M^{1/2} V_M^T}$

Now we have completed our objective, finding a decomposition of A with rotations V_Σ, V_M and scaling $\Lambda_M^{1/2}$.

Turns out you can construct M and Σ when A is not square. This is the grounds for <u>Singular Value Decomposition</u>.