Define: $A \in \mathbb{R}^{n\times m}$

$$\left.\begin{aligned}(m\times m)\quad A^TA &= V_a \Lambda_a V_a^T\\ (n\times n)\quad AA^T &= V_b \Lambda_b V_b^T\end{aligned}\right\}$$ Eigendecompositions of Gram matrices. Gram matrices are positive semidefinite, which implies: 1. The elements of the diaganol matrix (eigenvalues) of $\Lambda_{a/b}$ are ≥ 0

2. The eigenvectors are orthonormal, $V_{a/b}^{-1} = V_{a/b}^T$

In these notes, I will prove that the non-zero eigenvalues of A^TA and AA^T are equal. Later, I will also show some insightful detours.

Define the matrix $M := V_a^T A^T V_b$ $(m\times n)$

Calculate the following: $$M^TM = V_b^T A \cancel{V_a V_a^T} A^T V_b = V_b^T AA^T V_b = \cancel{V_b^T V_b}\, \Lambda_b \cancel{V_b^T V_b} = \Lambda_b \qquad (1)$$

$MM^T = \Lambda_a$ using the same logic as above.

Multiply both sides by $\Lambda_{a/b}^{-1/2}$ for both equalities.

$$\Lambda_b^{-1/2} M^T M \Lambda_b^{-1/2} = \Lambda_b^{-1/2} \Lambda_b \Lambda_b^{-1/2} = I^{(n\times n)}$$ and similarly $$\Lambda_a^{-1/2} M M^T \Lambda_a^{-1/2} = I^{(m\times m)}$$

Lemma: We can assume $m=n$ w.l.o.g., if $m<n$, b/c Λ_b has <u>at least</u> $n-m$ zeros on its diaganol.

Proof: If $m<n$, M is a wide matrix $\underbrace{\left[\;\equiv\;\right]}_{n}\Big\}m$, and therefore, $Null(M)$ is <u>at least</u> $n-m$ dimensions.

Consider the quadratic equation $x^TM^TMx = x^T\Lambda_b x$. $x \in \mathbb{R}^n$

If $\dim(Null(M)) \geq n-m$, there <u>must</u> exist $x \neq 0$ such that $x^T\Lambda_b x = 0$

This is <u>impossible</u> if Λ_b has only positive elements on its diaganol, so Λ_b must have at least $n-m$ zeros on the diaganol.

We can define an effective $\bar{M}, \bar{\Lambda}_b$, $$M = [\underbrace{\bar{M}}_{m}, \underbrace{0}_{n-m}] \qquad \underset{n\times n}{\Lambda_b} = \begin{bmatrix}\bar{\Lambda}_b & 0\\ 0 & 0\end{bmatrix} \quad \bar{\Lambda}_b:\ m\times m$$

we know this is a block of $m\times(n-m)$ zeros, it's the only way to get those zeros on the Λ_b diaganol.

$$\left.\begin{aligned} M^TM &= \begin{bmatrix}\bar{M}^T\\ 0\end{bmatrix}\begin{bmatrix}\bar{M} & 0\end{bmatrix} = \begin{bmatrix}\bar{\Lambda}_b & 0\\ 0 & 0\end{bmatrix}\\ MM^T &= \begin{bmatrix}\bar{M} & 0\end{bmatrix}\begin{bmatrix}\bar{M}^T\\ 0\end{bmatrix} = \bar{M}\bar{M}^T = \Lambda_a\end{aligned}\right\}$$

This demonstrates we can safely set $n=m$, and $M=\bar{M}$, $\Lambda_b = \bar{\Lambda}_b$.

Set $n=m$. Refer back to the equations derived above:

$$\left(\Delta_b^{-1/2} M^T\right)\left(M\Delta_b^{-1/2}\right) = I^{(m\times m)} \qquad \left(\Delta_a^{-1/2} M\right)\left(M^T \Delta_a^{-1/2}\right) = I^{(n\times n)} \qquad (2)$$

Define $N_a := M\Delta_b^{-1/2}$ $\qquad N_b := M^T\Delta_a^{-1/2}$

It is clear that $N_a^T N_a = I$, $N_b^T N_b = I$, so $N_{a/b}^T = N_{a/b}^{-1}$

$$N_a^T = N_a^{-1} \qquad\qquad N_b^T = N_b^{-1}$$

$$\Delta_a^{-1/2} M = \Delta_a^{1/2} M^{-T} \qquad\qquad \Delta_b^{-1/2} M^T = \Delta_b^{1/2} M^{-1}$$

multiply $\Delta_a^{1/2}$ $\qquad$ multiply $\Delta_b^{-1/2}$

$$M = \Delta_a M^{-T} \qquad\qquad M^{-1} = \Delta_b^{-1} M^T$$

take inverse

$$M^{-1} = M^T \Delta_a^{-1}$$

* Caution, we can only do the following steps with invertible square matrices, which we know is true of M, $\Delta_{a/b}$.
If there are any zeros on the diagonals of $\Delta_{a/b}$, we can safely truncate those rows/columns.

$\boxed{\text{so now ALL of } \operatorname{diag}(\Delta_{a/b}) > 0.}$

$$M^T\Delta_a^{-1} = \Delta_b^{-1} M^T$$

$$M^T = \Delta_b M^T \Delta_a^{-1} = \Delta_b^{-1} M^T \Delta_a$$

$$\Downarrow$$

$$\Delta_b^2 M^T = M^T \Delta_a^2 \qquad (3)$$

$$\Downarrow$$

$$\begin{bmatrix} & \vdots & \\ \cdots & b_i^2 m_{ij} & \cdots \\ & \vdots & \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & a_j^2 m_{ij} & \cdots \\ & \vdots & \end{bmatrix} \quad \text{for all } i,j \in \{1,\dots,n\}$$

Assume w.l.o.g. that $\quad a_1 \geq a_2 \geq \dots \geq a_n > 0$

$\qquad\qquad\qquad\qquad\quad b_1 \geq b_2 \geq \dots \geq b_n > 0$

$$\Delta_a = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix} \quad \Delta_b = \begin{bmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{bmatrix} \qquad (4)$$

The equality $b_i^2 m_{ij} = a_j^2 m_{ij}$ is true if either:

1. $m_{ij} = 0$
2. $b_i = a_j$

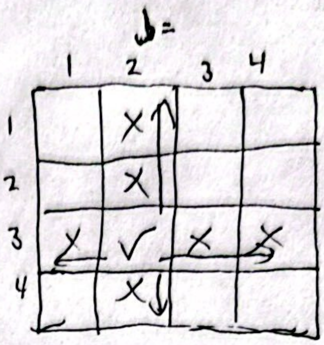
Furthermore, we know M, M^T are orthogonal from (1), so there is no full column-row that is just zeros.
So every row/column MUST have at least one non-zero element, and therefore $a_j = b_i$.

This can be illustrated with a picture.

Put a √ in a box if $b_i = a_j$

Put a X in a box if $m_{ij} = 0$

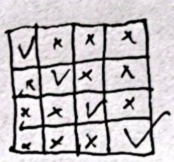
Each box must have either X or √.



$n = 4$

Putting √ at $i=3$, $j=2$ immediately puts X on the corresponding rows/columns, crossing out row 3 and column 2.

If we force the condition in (4), you get



$$\boxed{\therefore\ a_i = b_j} \Downarrow \boxed{\Delta_a = \Delta_b}$$

with repeated eigenvalues it still works.

QED

However, the story isn't finished. Where did I get the inspiration to define M? (1)

The expressions A^TAA^TA and AA^TAA^T are tempting, because you exploit associativity.

$$A(A^TA)A^T = (AA^T)(AA^T)$$

$$\Downarrow \text{ eventually}$$

$$V_b^T A V_a \Delta_a \underbrace{V_a^T A^T V_b}_{M} = \Delta_b^2 \quad (5)$$

This is where I found M.

Note: If you canNOT set $n=m$ like I did in the proof, you can get a proof by contradiction. (and M is NOT diagonal)

Sketch: Since $M^TM = \Delta_b$, $\exists\, U$ s.t. U^TU, $\tilde{M} = UM = \begin{bmatrix} \sqrt{b_1} & & 0 \\ & \sqrt{b_2} & \\ 0 & & \ddots \end{bmatrix}$ $m \times n$

Using a whitening transform on (5)

$$\left.\begin{aligned} (\tilde{M}^T U)\Delta_a(U^T\tilde{M}) &= \Delta_b^2 \\ \tilde{M}^T\tilde{M} &= \Delta_b \end{aligned}\right\} \rightarrow$$

exercise to reader: for $n \neq m$, M not diag, show why these two equations are impossible.

I wasted the most time trying to reconcile these two equations, let that be a lesson.