Gamma function

① $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

② $1 = e^{-x} + xe^{-x} + \frac{1}{2}x^2 e^{-x} + \ldots = e^{-x} \sum_{n=0}^{\wedge} \frac{x^n}{n!}$

③ Integrate both sides ~~multiple times~~

$$\int 1\,dx = \int e^{-x}dx + \int xe^{-x}dx + \int \tfrac{1}{2}x^2 e^{-x}dx + \ldots$$

$$x = -e^{-x}\left[(x+1) + \left(\tfrac{1}{2}x^2 + x + 1\right) + \left(\tfrac{1}{6}x^3 + \tfrac{1}{2}x^2 + x + 1\right) + \ldots\right]$$

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -e^{-x}(x+1)$$

$da = e^{-x}dx \quad b = x$

$a = -e^{-x} \quad db = dx$

$$\int \tfrac{1}{2}x^2 e^{-x}dx = -e^{-x}\left(\tfrac{1}{2}x^2 + x + 1\right)$$

$\downarrow$

$$\int \tfrac{1}{6}x^3 e^{-x}dx = -e^{-x}\left(\tfrac{1}{6}x^3 + \tfrac{1}{2}x^2 + x + 1\right)$$

$\downarrow$

$\vdots$

④ $\boxed{\lim_{n\to\infty} \int \frac{1}{n!}x^n e^{-x}dx = -e^{-x} \lim_{n\to\infty} \sum_{i=0}^{n} \frac{x^i}{i!} = -1}$

Last term in $\int 1dx$ Taylor series

Write expression for each term in $\int 1dx$ taylor series.

⑤ $\int \frac{1}{n!}x^n e^{-x}dx = -e^{-x} \sum_{i=0}^{n} \frac{x^i}{i!}$

Multiply both sides by $n!$

$$\int x^n e^{-x}dx = \left(-e^{-x} \sum_{i=0}^{n} \frac{x^i}{i!}\right) n!$$

We can isolate $n!$ by evaluating the indefinite integral expression at different points.

⑥

$$-e^{-x} \sum_{i=0}^{n} \frac{x^i}{i!}\Bigg|_{x=0} = -1$$

$$-e^{-x} \sum_{i=0}^{n} \frac{x^i}{i!}\Bigg|_{x\to\infty} = 0$$

⑦ $\int_0^\infty x^n e^{-x}dx = n!\left[-e^{-x}\sum_{i=0}^{n}\frac{x^i}{i!}\right]_0^\infty = n!\left[0-(-1)\right] = n!$

$$\boxed{n! = \int_0^\infty x^n e^{-x}dx}$$