The Nature of Line Integrals in 2D

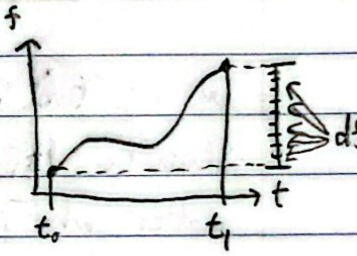
From the fundamental theorem of calculus, we know that

$$\int_{t_0}^{t_1} \frac{df}{dt}\, dt = f(t_1) - f(t_0) \quad (1)$$

This theorem can be used two ways. First, if you want to find the area under a curve, all you need to do is take the difference between two points of the antiderivative. The second is the opposite direction: If you want to find the difference between two points, you just need to integrate the derivative.

The intuition behind the second approach can be seen if we write

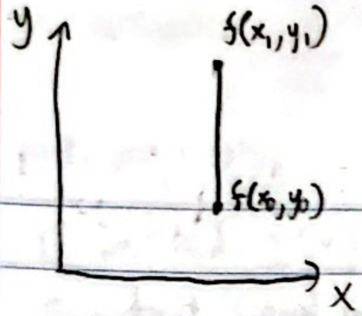
$\int_{t_0}^{t_1} \frac{df}{dt}\, dt$ as $\int_{f(t_0)}^{f(t_1)} df$. If the function f looks like



The height difference between the points t_1 and t_0 is simply the accumulated stack of df, hence the integral $\int df$.

With this knowledge, we could ask ourselves "How do we find a height difference between two points on a surface?"

In other words, find $f(x_1, y_1) - f(x_0, y_0)$.

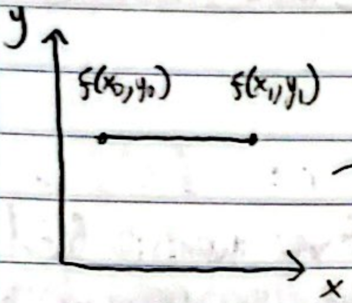


If we simplify the problem and make the two points vertical or horizontal to one another, then we can just take the same integral we took before.

Vertical: $\int_{y_0}^{y_1} \partial_y f(x_0,y)\,dy$

Notation
$\partial_y f(x_0,y) = \frac{\partial f}{\partial y}(x_0,y)$

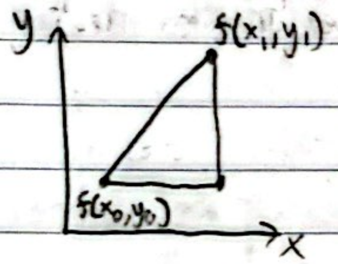
2



Horizontal: $\int_{x_0}^{x_1} \partial_x f(x,y_0)\,dx$

$\partial_x f(x_0,y) = \frac{\partial f}{\partial x}(x_0,y)$

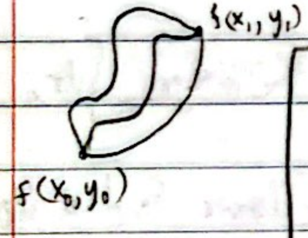
Now, if we combine them both...



$$f(x_1,y_1) - f(x_0,y_0) = \int_{x_0}^{x_1} \partial_x f(x,y_0)\,dx + \int_{y_0}^{y_1} \partial_y f(x_0,y)\,dy$$

3.1

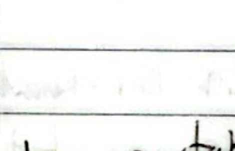
We can find the difference for any $f(x_0,y_0)$ and $f(x_1,y_1)$.
Since the difference between any two points is constant, it does not matter what path we use to integrate.



This is why the curl of a gradient is always 0.

$\nabla \times \nabla f = 0$, where $\nabla f = \begin{bmatrix} \partial_x f \\ \partial_y f \end{bmatrix}$

If you take a path and you end up where you started, your total change in value is 0.



To write more general expressions of the curves that we integrate over, we need to use parametric functions.

Let $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ $\quad dr = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$

3.2

Then $f(x_1,y_1) - f(x_0,y_0) = \int_{t_0}^{t_f} \partial_x f(x(t),y(t)) \frac{dx}{dt}\,dt + \int_{t_0}^{t_f} \partial_y f(x(t),y(t)) \frac{dy}{dt}\,dt = \boxed{\int_{t_0}^{t_f} (\nabla f \cdot dr)\,dt}$

Volume under partial differential

Notice how there is a general theme of mapping 2D objects (areas) to 1D objects (lengths).

$$f_1 - f_0 = \int f'$$

$$1D \leftarrow 2D$$

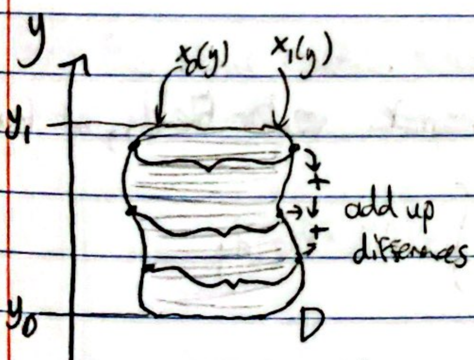
Perhaps this means we can map 3D volumes to a difference between 2D areas. This can continue on for more dimensions, which is the crux of Generalized Stokes Theorem.

Since we've been taking the areas under the functions $\partial_x f$ and $\partial_y f$, let's see what happens when we take volumes under them. We can do this because $f(x,y)$, $\partial_x f(x,y)$, $\partial_y f(x,y)$ are all surfaces.

$$\iint_D \partial_x f(x,y)\, dA = \int_{y_0}^{y_1}\int_{x_0(y)}^{x_1(y)} \partial_x f(x,y)\, dx\, dy = \int_{y_0}^{y_1} f(x_1(y), y) - f(x_0(y), y)\, dy$$

* Just worry about $\partial_x f$ for now.

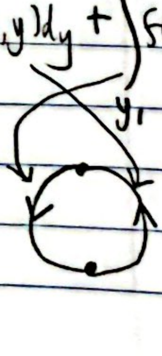
We've seen this inner integral before when we were trying to calculate $f(x_1, y_1) - f(x_0, y_0)$. Check 2



One way to interpret this is that we accumulate all the differences along the boundaries, therefore, we are subtracting two areas like we expected. However, if we split the integral:

$$\int_{y_0}^{y_1} f(x_1(y), y)\, dy - \int_{y_0}^{y_1} f(x_0(y), y)\, dy = \int_{y_0}^{y_1} f(x_1(y), y)\, dy + \int_{y_1}^{y_0} f(x_0(y), y)\, dy$$

You can see that we are integrating $f(x,y)$ counterclockwise, assuming the area is closed by a smooth loop.



$$\iint_D \partial_x f \, dA = \int_{y_0}^{y_1} f(x_1(y), y)\,dy + \int_{y_1}^{y_0} f(x_0(y), y)\,dy$$

y_1, y_0 counterclockwise

(4)

$$\iint_D \partial_y f \, dA = \int_{x_0}^{x_1} f(x, y_1(x))\,dx + \int_{x_1}^{x_0} f(x, y_0(x))\,dx$$

clockwise

Since $\partial_x f$ and $\partial_y f$ inherently go counterclockwise and clockwise respectively, then the integrals cancel out, $\iint_D \partial_x f + \partial_y f \, dA = 0 \Rightarrow \iint_D \|\nabla f\|_1 \, dA = 0$

dubious??

Areas under $f(x,y)$

There is a small detour we can take. So far we've been taking the integrals of differentials, we can also work out an expression for $\int_{(x_0 y_0)}^{(x_1 y_1)} f(x,y)\,ds$. Again, we must parameterize.

Also, from now on, $q = (x, y)$, so $f(x,y) = f(q)$

$$\int_{t_0}^{t_1} f(q(t)) \frac{ds}{dt}\,dt = \int_{t_0}^{t_1} f(q(t)) \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt \quad (5)$$

proof for this is actually longer than expected, $s\dot{s} = x\dot{x} + y\dot{y}$ not hard though.

Keep in mind, this is NOT how you integrate vector fields, at least normally.

$$\int_{t_0}^{t_1} [f_1(q(t)) + f_2(q(t))] \frac{ds}{dt}\,dt \neq \int f_1(q)\,dx + f_2(q)\,dy$$

Line Integrals of vector fields

horizontal, cosine, dot — $\nabla \cdot F$ vertical, sine, det — $\nabla \times F$

Define a vector field $F = \begin{bmatrix} f_1(q) \\ f_2(q) \end{bmatrix}$. It's Jacobian is $\begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix}$

Since we have expressions for individual areas under partial derivatives (4), we can also look for similar expressions for divergence and curl.

(6) $$\iint_D \nabla \cdot F \, dA = \iint_D \partial_x f_1 + \partial_y f_2 \, dA = \oint_y f_1(q)\,dy \overset{(\star)}{-} \oint_x f_2(q)\,dx = \int_{t_0}^{t_1} f_1(q(t))\frac{dy}{dt} - f_2(q(t))\frac{dx}{dt}\,dt$$

$$= \int_{t_0}^{t_1} (F \cdot n)\,dt$$ <u>Divergence Theorem</u>

(7) $$\iint_D \nabla \times F \, dA = \iint_D \partial_x f_2 - \partial_y f_1 \, dA = \oint_y f_2(q)\,dy \overset{(\star)}{+} \oint_x f_1(q)\,dx = \int_{t_0}^{t_1} f_1(q(t))\frac{dx}{dt} + f_2(q(t))\frac{dy}{dt}\,dt$$

$$= \int_{t_0}^{t_1} (F \cdot dr)\,dt$$ <u>Green's Theorem</u>

(★) The signs switch because the x integral goes clockwise.

Cauchy Integral Theorem

Complex functions can be treated as vector fields, where

$$f(z) = u(x,y) + i\,v(x,y)$$

In order for a complex function to be differentiable, it's derivative at all points must take the form of a complex number. Therefore, it's Jacobian must take the 2x2 complex number form.

$$\begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} \Rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

This leads to the Cauchy Riemann equations:

$$\partial_x u = \partial_y v \,, \quad \partial_x v = -\partial_y u \quad (8)$$

From equation (7), the loop integral is equal to

$$\iint_{dA} \partial_x v - \partial_y u \, dA = \oint_t f(z) \frac{dz}{dt} dt$$

It might be tempting to use (8) and say:

$$\iint_{dA} \partial_x v - \partial_y u \, dA = 2 \iint_{dA} \partial_x v \, dA$$

but this doesn't take into account that ∂_x and ∂_y go different orientations.

8 tells us that $\iint \partial_x v$ and $\iint \partial_y u$ will have the same magnitude, but the directions of the derivatives tells us they will go opposite orientations and cancel out. Therefore,

$$\oint f(z) dz = 0 \,, \quad \text{Cauchy Integral Theorem}$$