Line-Volume Integrals Part 2

This was written in Obsidian, which has embedded LaTex.

In this section, I ask and answer two questions:

1. Can we translate line integrals to surface integrals?
2. What does the differential form mean in terms of a Taylor Series approximation?

Surface Translation

1-dim Review

First, a review of Taylor and Leibniz rule relationship in 1-D.
From Fundamental Theorem of Calculus and the definition of a Taylor Series:
($f^{(i)}(x)$ is i-th derivative)

$$f(a) - f(0) = \int_0^a \frac{df}{dx}(x)dx = \sum_{i=1}^{\infty} \frac{f^{(i)}(0)a^n}{n!}$$

For very small a, both Taylor expansion and Leibniz rule give you the Euler Approximation:

$$f(a) = f(0) + \frac{df}{dx}(0)a$$

2-dim Case, Find Surface Function

Now consider the 2D case.

Define $\Delta f = f(a,b) - f(0,0) = \int_0^a \partial_x f(x,0)dx + \int_0^b \partial_y f(a,y)dy$
Define $df = \Delta f$ when the integration bound (a,b) is very small.

From the lecture notes "Line-Volume Integrals Part 1", we already know that there exist an infinite number of integration paths between $(0,0)$ and (a,b) that will give you the same result Δf. This also means there are an infinite number of surface functions $s(x,y)$ such that:
$$\Delta f = \int_0^a \int_0^b s(x,y)dxdy$$

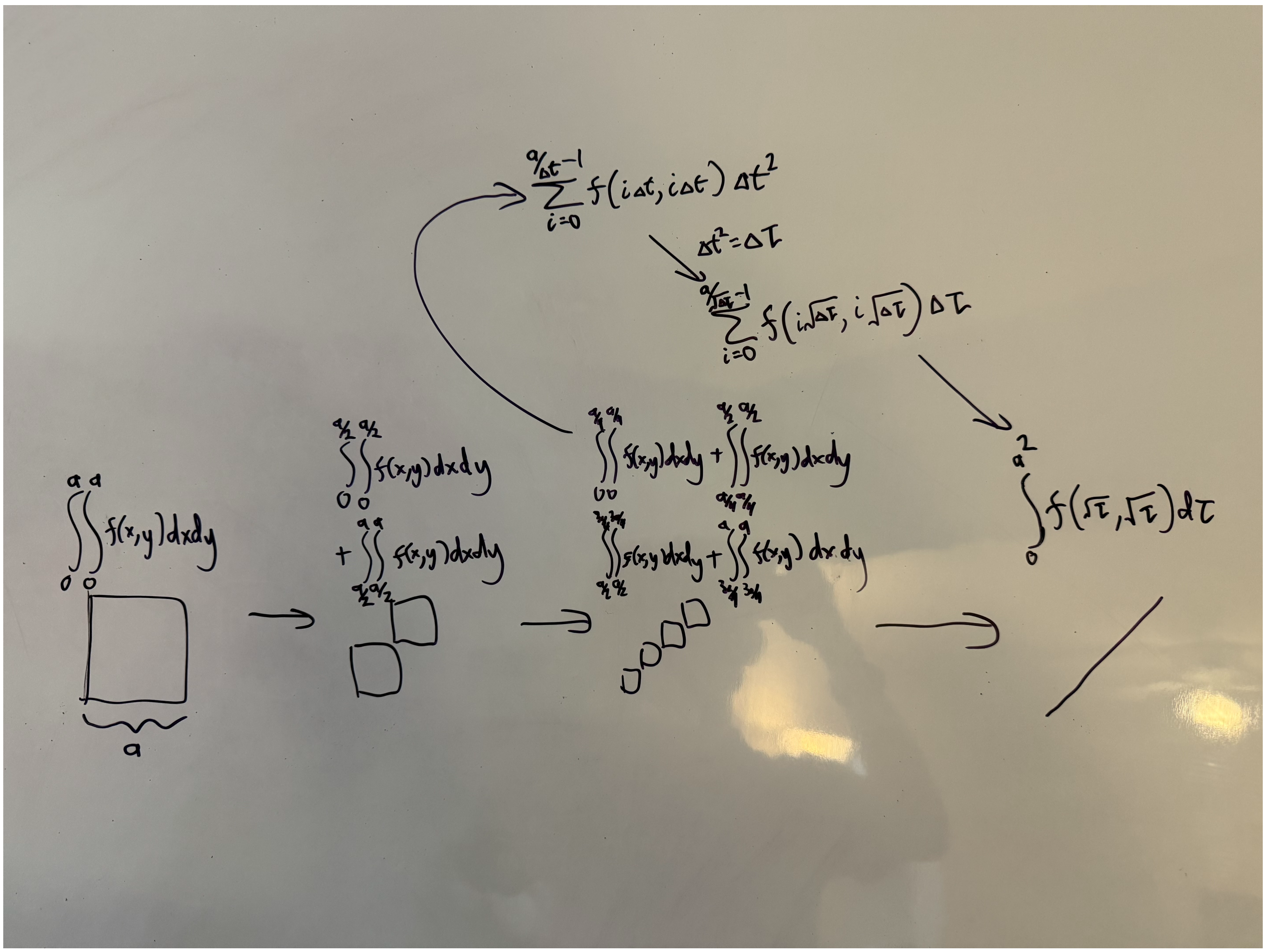
Two candidates for $s(x,y)$ include:
$s(x,y) = \frac{\partial_x f(x,0)}{b} + \frac{\partial_y f(a,y)}{a}$ (integrate horizontally first, then vertically)
$s(x,y) = \frac{\partial_x f(x,b)}{b} + \frac{\partial_y f(0,y)}{a}$ (vice-versa)
For the above examples, we are exploiting the fact that $\int_0^a dx = a$. We can do a "multiple by

one" trick inside the integral to get a surface integral, like this:

$$\int_0^b \partial_y f(0,y)dy = \frac{1}{a}\int_0^a dx \int_0^b \partial_y f(0,y)dy = \int_0^a \int_0^b \frac{\partial_y f(0,y)}{a} dy$$

The rest can be constructed by using piecewise variations of this idea.

Is there a continuous generalization of this? Consider the following exercise. Due to staircase paradox (and also just an incorrect infinitesimal translation), the last step where the sum transforms into a smooth 1-dim integral is not valid, and this smooth 1-dim integral most likely does not exist.



NOTE: This is a very different surface compared to the one considered in Part 1, which was $s(x,y) = \partial_x f + \partial_y f$. The antiderivative is different, and you get Divergence/Green's Theorem out of this.

Per the Leibniz approximation (multiple ways to get this, you can parameterize with $x(t), y(t)$, but you actually don't have to), we get, using the definition of dr and ∇f from above:

$$df = \partial_x f(0,0)a + \partial_y f(0,0)b = \nabla f^\top dr = \frac{s(0,0)}{ab}ab$$

The last part, writing it in terms of a surface, goes to show that trying to find a surface that gives you the difference is degenerate. The surface function itself is a function of the terms a and b. A "proper" Leibniz approx. of a surface integral would not have this dependency, although it still works out in the end.

Since dr can be any orientation, the stationarity condition is $\nabla f = 0$.

Relationship with area equations and surfaces

Assume a function for area is $A(x, y)$, which is actually a surface function.
Notice that the dA used in integration $\int dA$ is not the same as the derivative of the first-order Taylor term $dA = \partial_x A dx + \partial_y A dy$, since one is a 1-form and the other is a 2-form.

Instead, the dA used integration seems pick from the second order Taylor term $\partial_{xy} A$, which makes sense because it cancels with $dxdy$ (this is how you get $dA = r dr d\theta$). When integrating a general $s(x, y)$, second-order and cross-terms cannot simply be cancelled out. In other words, you must know how the function changes with x-y TOGETHER, not separately as in the first order.

$A(x, y) = xy$ and $A(r, \theta) = 0.5\theta r^2$ are the result of integrals (antiderivatives) with specific integration bounds, shouldn't be seen as "surface functions" that can be integrated. You can demonstrate this contradiction by taking in the fact that their 1-form differentials are nothing like the 2-form. So dA_{scalar} is a different object than dA in general.
Safer to do $\omega = dx \wedge dy$ and $A(x, y) = \int_0^x \int_0^y \omega$

Higher-Order Taylor

1-dim review

$f(a) = f(0) + \frac{d}{dx} f(0) a + \frac{1}{2} \frac{d^2}{dx^2} f(0) a^2 + \ldots$
Take derivative w.r.t. a: $\frac{d}{dx} f(a) = \frac{d}{dx} f(0) + \frac{d^2}{dx^2} f(0) a + \ldots$
We get the second term by considering the small difference: $\frac{d}{dx} f(a) - \frac{d}{dx} f(0)$

2-dim case

Getting the next Taylor Term requires us to consider the surface for small (a, b)
$(s(a, b) - s(0, 0))$ w.r.t. very small area $A = ab$, which means the differences $\partial_x(a, b) - \partial_x(0, 0)$ and $\partial_y(a, b) - \partial_y(0, 0)$. We use the result of the Leibniz approximation $df = \partial_x f(0, 0) a + \partial_y f(0, 0) b$ from the previous section, which ends up being the 1-st differential form

Expand and multiply by: $ds = \partial_x(\partial_x f/b + \partial_y f/a) a + \partial_y(\partial_x f/b + \partial_y f/a) b$
$= \partial_x^2 f \frac{a}{b} + \partial_y^2 f \frac{b}{a} + 2\partial_{xy} f$
We multiply by the small area ab to get the second order Taylor (since $df = \dfrac{s(a, b)}{ab} ab$):
$\frac{1}{2}(\partial_x^2 f(0, 0) a^2 + \partial_y^2 f(0, 0) b^2 + 2\partial_{xy} f(0, 0) ab)$

This checks out. By the same reasoning, you binomial theorem your way through the rest of the terms.