A convoluted way of characterizing discrete, 1D, linear random processes

normal distribution

Given a system $x_{t+1} = a x_t + w_t$, $w_t \sim N(0, \sigma)$, how do we describe $p(x_t)$?

$x_t \in \mathbb{R}$ $a \in \mathbb{R}$ $w \in \mathbb{R}$ $\sigma \in \mathbb{R}$ $\quad$ $\underline{x_0 \text{ is a constant}}$

There is a very simple way to characterize $p(x_t)$ that can be done in a few lines. This is not the purpose of this lecture. Instead, I will derive this result by first considering the joint distribution $p(x_t, x_{t-1}, \dots x_1, x_0)$.

Since the process is Markov: $p(x_t \dots x_0) = p(x_t|x_{t-1})\,p(x_{t-1}|x_{t-2}) \dots p(x_2|x_1)\,p(x_1|x_0)\,p(x_0)$

$$= \prod_{i=0}^{t-1} p(x_{i+1}|x_i)\; p(x_0)$$

Since $w_t \sim N(0,\sigma)$, we know $p(x_{i+1}|x_i) = N(ax_i, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_{i+1}-ax_i)^2}{2\sigma^2}\right)$

So $p(x_t \dots x_0) = \prod_{i=0}^{t-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_{i+1}-ax_i)^2}{2\sigma^2}\right) p(x_0) = \frac{1}{(2\pi\sigma^2)^{t/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=0}^{t-1}(x_{i+1}-ax_i)^2\right) p(x_0)$

In order to find $p(x_t)$, we can integrate over all the variables $x_0 \dots x_{t-1}$

$$p(x_t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_{t-1}\, dx_{t-2} \dots dx_2\, dx_1\, dx_0\; p(x_t \dots x_0)$$

Since x_0 is constant, we can get rid of ~~[illegible]~~ the integral for dx_0, since $\int_{-\infty}^{\infty} dx_0\, p(x_0) = 1$. Note: If x_0 were not constant, the result is the same ~~[illegible]~~ but shifted over by one time step. Keeping x_0 constant makes things slightly easier.

So now we have:

$$p(x_t) = \frac{1}{(2\pi\sigma^2)^{t/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_{t-1} \dots dx_1 \exp\left(-\frac{1}{2\sigma^2}\sum_{i=0}^{t-1}(x_{i+1}-ax_i)^2\right)$$

The key to figuring out this integral is just focus on the sum $\sum_{i=1}^{t-1}(x_{i+1}-ax_i)^2$. Let's organize it as such:

Find $p(x_2)$: $(x_1-ax_0)^2 + (x_2-ax_1)^2 \longleftarrow$ Integrate over x_1

Find $p(x_3)$: $(x_3-ax_2)^2 \longleftarrow$ Integrate over x_2

Find $p(x_4)$: $(x_4-ax_3)^2 \longleftarrow$ Integrate over x_3

$\vdots \qquad\qquad \vdots$

Find $p(x_{t-1})$: $(x_{t-1}-ax_{t-2})^2 \longleftarrow$ Integrate over x_{t-2}

Find $p(x_t)$: $(x_t-ax_{t-1})^2 \longleftarrow$ Integrate over x_{t-1}

Because terms later in time are dependent on previous terms, we will first integrate over x_1, and then iterate up until reaching x_{t-1}.

To integrate over x_1, we must ~~[illegible]~~ gather all terms with x_1, so we start with $(x_1-ax_0)^2 + (x_2-ax_1)^2$.

We know that $\boxed{\int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \sqrt{2\pi\sigma^2}}$, so if we can fit the expression $(x_1-ax_0)^2 + (x_2-ax_1)^2$ into ~~[illegible]~~ some form $\frac{(x_1-\mu)^2}{\alpha}$, then we can find the integral right away as $\sqrt{2\pi\sigma^2\alpha}$.

With some manipulation and completing the square, we get:

$$(x_2-ax_1)^2+(x_1-ax_0)^2 = (a^2+1)\left(x_1-\frac{a(x_2+x_0)}{a^2+1}\right)^2+\frac{1}{a^2+1}(x_2-a^2x_0)^2 \quad ①$$

This successfully separates out the x_1 terms and puts it in the desired form, so we get

$$p(x_2)=\frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}dx_1\, \exp\left(\frac{(a^2+1)}{2\sigma^2}\left(x_1-\frac{a(x_2+x_0)}{a^2+1}\right)^2\right)\exp\left(-\frac{1}{2\sigma^2(a^2+1)}(x_2-a^2x_0)^2\right) = \frac{\sqrt{2\pi\sigma^2/a^2+1}}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2(a^2+1)}(x_2-a^2x_0)^2\right)$$

$$=\frac{1}{\sqrt{2\pi\sigma^2(a^2+1)}}\exp\left(-\frac{1}{2\sigma^2(a^2+1)}(x_2-a^2x_0)^2\right)=\mathcal{N}\left(a^2x_0,\sqrt{a^2+1}\,\sigma\right)$$

The term (a^2+1), which is important for finding the integral and variance of $p(x_2)$, is the coefficient of x_1^2.

Let's generalize ① as $$\alpha_1(x_1-\xi_1)^2+\frac{1}{P_1}(x_2-a^2x_0)^2 = k_1^1+k_2^1 \quad ②$$

(ξ_i is just a term that accumulates all the terms that will be cancelled after integration)

$\Downarrow$ generalize

$$\alpha_i(x_i-\xi_i)^2+\frac{1}{P_i}(x_{i+1}-a^{i+1}x_0)^2 = k_1^i+k_2^i \quad ③$$

Any k_1^i can be integrated away over x_i to produce the term $\sqrt{\frac{2\pi\sigma^2}{\alpha_i}}$. However, the term k_2^i remains as residuals that will pass on to the next integration over x_{i+1}.

To integrate over x_2, we must gather the x_2 terms:

$$k_2^1+(x_3-ax_2)^2 = \frac{1}{P_1}(x_2-a^2x_0)^2+(x_3-ax_2)^2$$

If we re-arrange these terms, we can get:

$$\left(a^2+\frac{1}{P_1}\right)(x_2-\xi_2)^2+\frac{1}{\alpha_2P_1}(x_3-a^3x_0)^2 = k_1^2+k_2^2$$

This is a very similar form to the $k_1^1+k_2^1$ equation we got before in ②.

We can see that α_i and P_i follow an update rule such that

$$\boxed{P_i=\alpha_iP_{i-1} \qquad \alpha_1=P_1=a^2+1 \qquad \alpha_i=a^2+\frac{1}{P_{i-1}}}$$

P_i can also be thought of as the product of all α_i, so

$$P_i=\prod_{j=1}^{i}\alpha_j$$

When $\alpha_1=a^2+1$, it turns out that $\boxed{P_i=\sum_{j=0}^{i}a^{2j}}$

With these update rules, we can find $p(x_t)$

$$p(x_t)=\frac{(2\pi\sigma^2)^{t-1/2}P_{t-1}^{1/2}}{(2\pi\sigma^2)^{t/2}}\exp\left(-\frac{1}{2\sigma^2P_{t-1}}(x_t-a^tx_0)^2\right)=\frac{1}{\sqrt{2\pi\sigma^2\sum_{i=0}^{t-1}a^{2i}}}\exp\left(-\frac{1}{2\sigma^2\sum_{i=0}^{t-1}a^{2i}}(x_t-a^tx_0)^2\right)=\boxed{\mathcal{N}\left(a^tx_0,\sigma\sqrt{\sum_{i=0}^{t-1}a^{2i}}\right)}$$

Note that when $a=1$, $p(x_t)=\mathcal{N}(x_0,\sqrt{t}\,\sigma)$, which is a well-known result in both continuous and discrete systems.